PMM U.S.S.R.,Vol.49,No.5,pp. 667-670,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

A METHOD OF STUDYING THE EXCITATION OF STEADY HARMONIC OSCILLATIONS IN A COMPOSITE, WEDGE-LIKE REGION*

A.A. LYAPIN and M.G. SELEZNEV

An approach is described, enabling the stress-deformation state of a composite elastic wedge to be studied in the mode of steady harmonic oscillations, when oscillating stresses distributed in some region act on the edge of the wedge. The solution of the problem is constructed for every wedge in the form of a superposition of solutions of the problems for elastic half-spaces whose boundaries intersect at the required angle. The application of the proposed method is illustrated by solving a model problem of antiplane oscillations of a composite elastic wedge. Some numerical results are given. The method makes it possible to use a unified approach to solving problems in a more general (two- and three-dimensional) formulation.

1. Let us consider the boundary value problem of the theory of elasticity of the steady oscillations of a composite wedge under an antiplane load. Let the elastic medium occupy the following region in the Cartesian rectangular system of coordinates x, y, z

$$-\infty < z < \infty \quad (x, y) \in D_j \quad (j = 1, 2)$$
$$D_1: \{y \geqslant 0,\ 0 \leqslant x \leqslant y \operatorname{tg} \beta_1\}, \quad D_2: \{y \geqslant 0,\ -y \operatorname{tg} \beta_2 \leqslant x \leqslant 0\}$$

The elastic properties of the medium in region D_j are defined by the density ρ_j and Lamé constants λ_j, μ_j. The motion of the points of the medium is described, in the case of harmonic antiplane oscillations of frequency ω, by an equation of the form

$$\Delta w_j + \theta_j^2 w_j = 0; \ \theta_j^2 = \rho_j \omega^2/\mu_j$$

(w_j is the amplitude function of the displacement of the point of the medium in the region D_j; $W_j(x, y, t) = w_j(x, y) e^{-i\omega t}$, $j = 1, 2$).

The following oscillatory stresses are specified at the boundary of the region:

$$x = -y \operatorname{tg} \beta_2, \ \tau_{zn} = T_3(y_3) e^{-i\omega t}; \ y_3 = -y/\cos\beta_2$$
$$x = y \operatorname{tg} \beta_1, \ \tau_{zn} = T_1(y_1) e^{-i\omega t}; \ y_1 = -y/\cos\beta_1 \tag{1.1}$$

(n is the normal to the plane $x = -y \operatorname{tg} \beta_2$ or $x = y \operatorname{tg} \beta_1$).

The conditions of rigid coupling are specified at the boundary separating the regions D_1 and D_2, and the stress and deformation tensor components tned to zero at infinity.

The solution of the problem of the oscillation of each wedge which is a part of the composite wedge, is constructed using the superposition principle, in the form of a sum of solutions to the problems of the steady oscillations of two half-spaces whose surfaces intersect at an angle β_1 or β_2. The boundary of each half-space is acted upon by, generally speaking, unknown, shear forces oriented along the z axis.

Let us consider a wedge-like region formed by the intersection of two half-spaces. Let the following shear forces be given at the edges of the wedge:

$$x = 0, \ \tau_{zx} = T_2(y) e^{-i\omega t}; \ x = y \operatorname{tg} \beta_1, \ \tau_{zn} = T_1(y_1) e^{-i\omega t}$$

Using the superposition method we arrive at the following expression describing the amplitude function of displacement of a point of the medium:

$$w_1(X, Y) = \frac{h}{2\pi\mu_1} \int_\sigma (\xi_1 + \xi_2)\, d\alpha \tag{1.2}$$

$$\xi_n = \int_{-\infty}^{\infty} Z_n e^{i\alpha\eta}\, d\eta \cdot k_n(\alpha, X, Y), \quad n = 1, 2; \quad k_1(\alpha, X, Y) = k_2(\alpha, X, Y)|_{\beta_1 = \pi}$$

$$k_2(\alpha, X, Y) = \gamma_1^{-1} \exp\{-\gamma_1(-X\cos\beta_1 + Y\sin\beta_1) + i\alpha(X\sin\beta_1 + Y\cos\beta_1)\}$$

$$\gamma_1 = (\alpha^2 - \rho_1\omega^2 h^2/\mu_1)^{1/2}, \ X = x/h, \ Y = y/h$$

The contour σ is determined by applying the principle of limit absorption, and has the following form /1/: it passes around the positive singularities of the integrand from below, and the negative ones from above, and the rest coincides with the real axis, and h is a constant with dimensions of length.

*Prikl.Matem.Mekhan.,49,5,867-871,1985

We find the functions $Z_1(Y)$, $Z_2(Y)$ from the solution of the following system of integral equations:

$$Z_1(-Y) - I_2^- = T_1(-Y),\ Z_2(Y) - I_1^+ = T_2(Y) \tag{1.3}$$
$$I_r^{\pm} = \frac{1}{2\pi}\int_0^\infty \bar{Z}_n(\alpha)\, k_2(\pm\alpha, 0, Y)(\gamma_1\cos\beta_1 \pm i\alpha\sin\beta_1)\gamma_1^{-1}\, d\alpha$$

($\bar{Z}_n(\alpha)$ are the Fourier transforms of $Z_n(Y)$).

Substituting the quantities characterizing the region D_2 we determine, in the same manner, the displalement field in the second wedge with elastic parameters ρ_2, μ_2.

Thus, using the proposed method, we determine the wave field in a composite body in two stages. In the first stage we must solve the following system of integral equations:

$$KZ = \mathbf{T} \tag{1.4}$$
$$Z = \operatorname{col}\{Z_1, \ldots, Z_4\},\ \mathbf{T} = \operatorname{col}\{T_1, 0, T_3, 0\}$$

for $Z_1(-Y), Z_2(Y), Z_3(-Y), Z_4(Y)$. The system is obtained from the conditions of contact (equality of the tangential stresses τ_{zx} and displacements on the common edge of the wedge) and the boundary conditions (1.1). In the second stage we determine the wave field in every wedge, substituting the function Z_j into (1.2).

We seek a solution of system (1.4) in the space of summable functions, and this ensures the finiteness of the energy of elastic oscillations in any bounded volume of the medium.

2. Let us inspect the behaviour of the functions Z_j sought near the angle point. We do it in order to construct an effective numerical scheme for solving system (1.4), taking into account the existence of a fixed singularity of the kernel of the integral operator of the system as $Y \to 0$ and the presence of a singularity, near the wedge apex, of the solutions of this system.

Replacing the value of the kernels in (1.4) by their limiting values as the parameter $\alpha \to \infty$ and retaining only the integral containing a singularity near the angle point, we apply to the system $K_0 \mathbf{Z}_0 = \mathbf{T}$ obtained in this manner the Mellin /2/ transform in Y.

As a result, the problem of finding $Z_0(Y)$ is reduced to computing an integral of the form

$$Z_{0j}(Y) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} Y^{-s}\frac{\Delta_j(s)}{\Delta(s)}\, ds$$

where

$$\Delta(s) = k\cos u_1 \sin u_2 + \sin u_1 \cos u_2 - \cos v \sin u_1 - k\cos v \sin u_2 + \tag{2.1}$$
$$k\cos v\cos u_1 + \sin v\cos u_2 - {}^1/_2 (1 + k)\sin 2v$$
$$u_n = 2(s - 1)(\pi - \beta_n),\ n = 1, 2;\ v = 2s\pi,\ k = \mu_1/\mu_2$$

Numerical analysis of the function (2.1) shows that there are no complex roots of the equation $\Delta(s) = 0$ within the strip $0 < \operatorname{Re} s < 1$. Fig.1 shows the dependence of the real roots s_k on the parameter β_2 when $\beta_1 = \pi/4$, $k = 1/2$. Having determined the order of the singularity of the function Z_j, we can construct an operator transforming the system (1.4) to a form suitable for numerical solution. We can use, as such an operator, one constructed earlier in the course of investigating the solutions of the system as $Y \to 0$, namely

$$K_0^{-1}\mathbf{R} = \frac{1}{2\pi i}\int_0^\infty \int_{\gamma - i\infty}^{\gamma + i\infty} k_0^{-1}(Y, \xi)\, \mathbf{R}(\eta)\, \xi^{\eta - 1}\, d\xi\, d\eta$$

When the operator $K_0^{-1}R$ acts on the system (1.4) from the left, we obtain, after a number of transformations, the following system of functions $X_j = Z_j(Y)\,Y^{s_*}$, regular in the space of summable functions, but without a singularity at the angle point:

$$\mathbf{X} + K_*\mathbf{X} = \mathbf{X} + \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty}\int_0^\infty k_0^{-1}(Y, s)\, s^{\eta - 1}\int_0^\infty \frac{k(\eta, \xi) - k_0(\eta, \xi)}{\xi^{s_*}}\,\mathbf{X}(\xi)\, d\xi\, d\eta\, ds \tag{2.2}$$
$$s_* = \max_k \{s_k\};\quad 0 < s_* < {}^1/_2$$

3. In what follows, we have used numerical methods to study the system. The Bubnov-Galerkin method /3/ was used to determine $Z_j(Y)$. Functions of the form $l_k(p, Y) = \exp(-pY)\, L_k(Y)$; $0 < p < 1$, $L_k(Y)$ are Lagger polynomials, were used as the basic system. The resulting infinite system of linear algebraic equations forthe coefficients of the expansions in l_k was studied using the reduction method /3/. We have found that retention of six terms in the expansion was sufficient to obtain a solution with an error not exceeding 10%. Analysis of the practical convergence of the reduction method showed that the most effective region from the point of view of numerical computations, is the region with a wedge-like cross-section lying in the half-plane. The convergence suffers when the shear modulus μ_1 of the half-space is greater than the shear modulus μ_2 of the wedge. Fig.2 shows the behaviour of the real and imaginary parts of the function Z_1 for $\mu_1 = 2\cdot 10^5$, $\mu_2 = 2\cdot 10^4$, $\beta_2 = \pi/6$, with an error not exceeding 10%.

4. In the second stage of the proposed method of solving the boundary value problem with help of the solution of system (1.4), we determined the wave field in the medium. The numerical results obtained enabled us to estimate the wedge displacement field when the parameters (β_1, β_2, ω, μ_1, μ_2) of the problem were varied.

We used as the basis the problem with a "soft" wedge in the half-space. Fig.3 shows the behaviour of the amplitude function of the displacement on moving away from the wedge tip into the half-space, for various values of β_2 (the solid, dash-dot, dash and dotted lines correspond to the values $\beta_2/\pi = {}^{11}/_{12}, {}^{5}/_{6}, {}^{3}/_{4}, {}^{2}/_{3}$), $\mu_1 = 2\cdot 10^5$, $\mu_2 = 10^5$; $X = Y$.

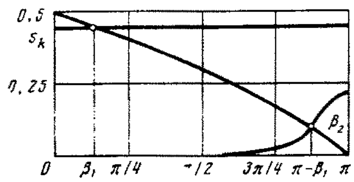


Fig.1

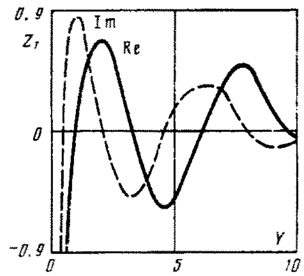


Fig.2

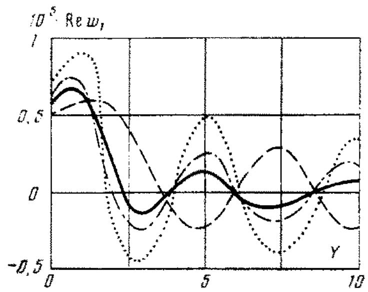


Fig.3

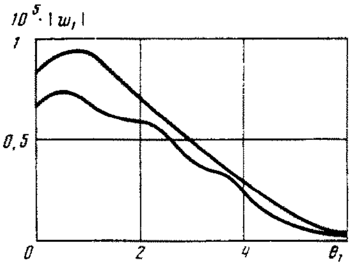


Fig.4

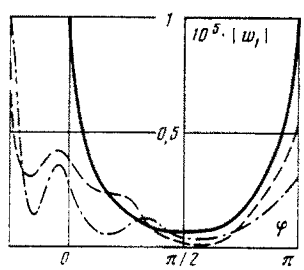


Fig.5

Fig.4 shows the amplitude-frequency characteristics of the point of the medium with coordinates $X = Y = 10$ for $\beta_1 = \pi$, $\beta_2 = \pi/4$, $\mu_1 = 2\cdot 10^5$, $\mu_2 = 10^5$. We see the correspondence between the behaviour of $w_1(\theta_1)$ and the amplitude-frequency characteristic of the same point of the half-space without a wedge. There are, however, frequency ranges determining the local amplitude growth. The ranges correspond to the neighbourhood of the "resonance" frequencies of the sputtered wedge. For $w_2(\theta_2)$ we have a displacement of the fundamental resonance frequency in the direction of lower frequencies, while the following frequencies are relatively stable.

Fig.5 gives a comparison of the diagrams of the displacement field for the half-space with and without the "soft" wedge, then the readius R of the circumference of the diagram is varied. The dashed line corresponds to $R = 12$, and the dash-dot line to $R = 20$, while the solid line corresponds to the half-space without a wedge for $R = 20$. In the region where the wedge is joined with the half-space, an oscillation is observed in the level of the modulus, as compared with a smoother pattern in the medium underneath the free surface. The shear wave propagating into the half-space has a lower intensity in the case of a region with a wedge. The energy redistribution caused by the wedge-like region is clearly seen. The increase in the rigidity μ_2 is accompanied by a natural strengthening of the influence of the wedge on the distribution of the displacement field in the half-space.

REFERENCES

1. VOROVICH I.I. and BABESHKO V.A., Mixed Dyhamic Problems of the Theory of Elasticity for Non-classical Domains. Moscow, Nauka, 1979.
2. UFLYAND YA.S., Integral Transforms in Problems of the Theory of Elasticity. Leningrad, Nauka, 1967

3. PARTON V.Z. and PERLIN P.I., Methods of the Mathematical Theory of Elasticity, Moscow, Nauka, 1981.

Translated by L.K.

PMM U.S.S.R.,Vol.49,No.5,pp.670-674,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

THE SCATTERING MATRIX IN A WAVEGUIDE WITH ELASTIC WALLS*

YU.A. LAVROV and V.D. LUK'YANOV

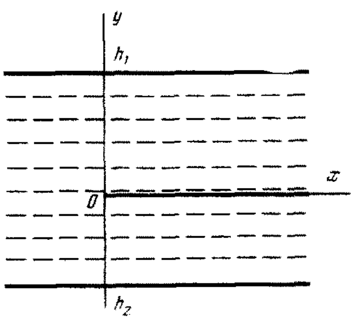
The spectrum of normal waves is studied and the scattering matrix is determined for a planar waveguide with elastic walls and with an elastic semi-infinite plate situated within it. The mechanical mode of behaviour of elastic plates is described using the general-type differential operators. Problems of this type belong to the class of the boundary contact problems /1, 2/. The unique solvability of these problems requires the formulation of additional boundary contact conditions describing the mechanical behaviour of the edge of the semi-infinite plate situated within the waveguide. The regularization of the integrals appearing when the general-type boundary contact conditions are satisfied is indicated.

1. Formulation of the problem.

We seek a solution of the following two-dimensional homogeneous Helmholtz equation:

$$\partial^2 P/\partial x^2 + \partial^2 P/\partial y^2 + k^2 P = 0 \tag{1.1}$$

in the strip $-\infty < x < +\infty$, $h_2 < y < h_1$ with a cut $y = 0, x > 0$ (see the figure), describing the distribution of the pressure $P(x, y)$ when the system is excited by a given acoustic field $P_0(x, y)$. Here $k = \omega/c$ is the wave number, ω is the angular frequency; here and henceforth the dependence of the wave processes on time, chosen here in the form $\exp(-i\omega t)$, is neglected; c is the velocity of sound in the medium.



The mechanical behaviour of the walls of the waveguide, i.e. of elastic plates, is described by the following boundary conditions:

$$L_j\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) P(x, h_j) = 0, \quad -\infty < x < +\infty, \quad j = 1, 2 \tag{1.2}$$

$$\left(L_j\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = (-1)^j M_{1j}\left(-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial y} + M_{2j}\left(-\frac{\partial^2}{\partial x^2}\right)\right)$$

A thin elastic plate is situated on the ray $y = 0, x > 0$ and it executes antisymmetric oscillations described by the boundary conditions $(x > 0)$

$$\frac{\partial P(x, +0)}{\partial y} = \frac{\partial P(x, -0)}{\partial y} \tag{1.3}$$

$$M_{13}\left(-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial y} P(x, 0) + M_{23}\left(-\frac{\partial^2}{\partial x^2}\right)[P(x, +0) - P(x, 0)] = 0 \tag{1.4}$$

Condition (1.3) describes the equality of the displacements of the upper (lower) surface of the plate $u(x) = (\rho_0\omega^2)^{-1}\partial P(x, \pm 0)/\partial y$, ρ_0 is the fluid density. We note that condition (1.3) holds on the continuation of the plate median $y = 0, x < 0$, as well as the condition thatt the pressure is continuous

$$P(x, +0) = P(x, -0), \quad x < 0 \tag{1.5}$$

Here $M_{1j}(-\partial^2/\partial x^2)$, $M_{2j}(-\partial^2/\partial x^2)$ $(j = 1, 2, 3)$ are polynomials whose coefficients depend on the mechanical properties of the elastic materials of which the waveguide walls are made.

We illustrate all this by describing the form of the differential operators for different types of the waveguide walls: $M_{1j} = 1$, $M_{2j} = 0$ (perfectly rigid walls); $M_{1j} = 0$, $M_{2j} = 1$ (perfectly pliable walls); $M_{1j} = \partial^2/\partial x^2 + K_j^2$, $M_{2j} = \rho_0\omega^2/N_j$ (elastic membranes); $M_{1j} = \partial^4/\partial x^4 - \varkappa_j^4$, $M_{2j} = \rho_0\omega^2/D_j$ (elastic, flexurally oscillating plates). Here K_j is the wave number of the waves within the membrane, $K_j = \rho_j/N_j$, ρ_j is the linear density of the membrane (plate), N_j is the tensile force in the membrane, $\varkappa_j$ is the wave number of the flexural waves in the plate situated in vacuo, and $\varkappa_j = \rho_j\omega^2/D_j$, D_j is the flexural rigidity of the plate.

---

*Prikl.Matem.Mekhan.,49,5,871-875,1985